Math 118B Final Practice Solutions

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1. In the compact metric space X a sequence of functions (f_n)—not necessarily continuous—converge pointwise to a continuous function f. Prove that the convergence is uniform if and only if for any convergent sequence $x_n \to x$ in X we have
$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

Proof. ($\Rightarrow$) Suppose $f_n \to f$ uniformly and let $x_n \to x$ in X. Let $\epsilon > 0$ and find N so that for all $n \geq N$ we have $\|f_n - f\|_\infty < \epsilon/2$. Since f is continuous and $x_n \to x$, we can enlarge N if necessary so that $|f(x_n) - f(x)| < \epsilon/2$ for all $n \geq N$. For all such n,
$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \leq \|f_n - f\|_\infty + \epsilon/2 < \epsilon.$$

Hence $f_n(x_n) \to f(x)$.

($\Leftarrow$) Suppose that f_n does not converge uniformly. Then there exists $\epsilon > 0$ and a subsequence x_{n_k} so that
$$|f_{n_k}(x_{n_k}) - f(x_{n_k})| \geq \epsilon$$
for all k. Passing to a convergent subsequence, assume that (x_{n_k}) converges to some point $x \in X$. Since f is continuous there exists $K > 0$ so that for all $k \geq K$ we have
$$|f(x_{n_k}) - f(x)| < \epsilon/2.$$
For all such k we have
$$|f_{n_k}(x_{n_k}) - f(x)| \geq |f_{n_k}(x_{n_k}) - f(x_{n_k})| - |f(x_{n_k}) - f(x)| \geq \epsilon/2.$$
Therefore if (x_n) is a sequence converging to x and containing (x_{n_k}) as a subsequence, then $f_n(x_n)$ does not converge to $f(x)$. □

2. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(2\pi\sqrt{n^2+x^2}\right)$ converges uniformly on bounded intervals.

Proof. Notice that for any $n \in \mathbb{N}$,
$$\sin^2\left(2\pi\sqrt{n^2+x^2}\right) = \sin^2\left(2\pi\sqrt{n^2+x^2} - 2\pi n\right) = \sin^2\left(\frac{2\pi x^2}{n} \cdot \frac{1}{\sqrt{1+(x/n)^2}+1}\right).$$
For positive numbers t we have $\sin t \leq t$, whence
$$\sin^2\left(2\pi\sqrt{n^2+x^2}\right) \leq \left(\frac{2\pi x^2}{n} \cdot \frac{1}{\sqrt{1+(x/n)^2}+1}\right)^2 \leq \frac{\pi^2 x^4}{n^2}.$$
If $|x| \leq M$, then the series converges uniformly by the Weierstrass M-test. □

3. Prove that the series $\displaystyle\sum_{n=1}^{\infty} n^2 x^2 e^{-n^2|x|}$ converges uniformly on $\mathbb{R}$.

Proof. Let $f_n(x) = n^2 x^2 e^{-n^2|x|}$. Since $f_n(0) = 0$, $f_n > 0$ on $(0, \infty)$, and $f_n \to 0$ as $x \to \infty$, there must be an absolute maximum on $[0, \infty)$. The maximum occurs when $f_n'(x) = 0$, that is, when

$$0 = \frac{d}{dx} \ln f_n(x) = \frac{2}{x} - n^2,$$

so that $f_n(2/n^2) = 4e^{-2}/n^2$ is the maximum on $[0, \infty)$. Since f_n is even, $|f_n| \le 4e^{-2}/n^2$ on $\mathbb{R}$. By the Weierstrass M-test, the series converges uniformly on $\mathbb{R}$. □

4. Determine the domain of convergence for $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{(-1)^n n^2} x^n.$

Proof. Denoting the coefficients by a_n, we have that

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = \limsup_{n\to\infty} \left(1 + \frac{1}{n}\right)^{(-1)^n n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n} = e.$$

By the Cauchy-Hadamard theorem, the radius of convergence is $1/e$. The values $x = \pm 1/e$ should be checked separately. □

5. Suppose that $f, g : [0, 1] \to \mathbb{R}$ are continuous. Prove there exists $c \in [0, 1]$ so that

$$\int_0^1 f(x)g(x)\,dx = f(c) \int_0^1 g(x)\,dx.$$

Proof. Since $[0, 1]$ is compact, we can find m, M so that $m \le f \le M$ everywhere on $[0, 1]$. Thus

$$m \int_0^1 g(x)\,dx \le \int_0^1 f(x)g(x)\,dx \le M \int_0^1 g(x)\,dx.$$

The problem statement is incomplete at this point; $g \ge 0$ must be assumed. If

$$\int_0^1 g(x)\,dx = 0$$

then $g \equiv 0$ (see problem 7 below), so the problem statement follows. Otherwise, define

$$y = \left(\int_0^1 g(x)\,dx\right)^{-1} \int_0^1 f(x)g(x)\,dx.$$

Since $m \le y \le M$ the intermediate value theorem guarantees $f(c) = y$ for some c in the domain of f. □

6. Fix $0 < a < b$ and a continuous function $f : \mathbb{R} \to \mathbb{R}$. Evaluate

$$\lim_{\epsilon\to 0^+} \int_{a\epsilon}^{b\epsilon} \frac{f(x)}{x}\,dx.$$

Proof. Given $\epsilon > 0$ find $c_\epsilon \in [a\epsilon, b\epsilon]$ so that

$$\int_{a\epsilon}^{b\epsilon} \frac{f(x)}{x}\,dx = f(c_\epsilon) \int_{a\epsilon}^{b\epsilon} \frac{1}{x}\,dx = f(c_\epsilon)\ln(b/a).$$

If $\epsilon \to 0$ then $c_\epsilon \to 0$ and $f(c_\epsilon) \to 0$ by the continuity of f. Hence

$$\lim_{\epsilon\to 0^+} \int_{a\epsilon}^{b\epsilon} \frac{f(x)}{x}\,dx = f(0)\ln(b/a).$$

□

7. Suppose that $f : [0,1] \to \mathbb{R}$ is continuous and $f \geq 0$. If

$$\int_0^1 f(x)\,dx = 0,$$

prove that f is identically 0.

Proof. If $f \not\equiv 0$ then for some $c \in (0,1)$ we have $f(c) > 0$. By continuity there exists a neighborhood $(c-\epsilon, c+\epsilon)$ upon which $f > f(c)/2$. Therefore

$$\int_0^1 f(x)\,dx \geq \int_{c-\epsilon}^{c+\epsilon} f(c)/2\,dx = \epsilon f(c) > 0.$$

□

8. Suppose that $f : [0,1] \to \mathbb{R}$ is continuous and that for each $n \geq 0$,

$$\int_0^1 x^n f(x)\,dx = 0.$$

Prove that f is identically 0.

Proof. By linearity of the integral we see that

$$\int_0^1 p(x)f(x)\,dx = 0$$

for any polynomial p. Using the Weierstrass approximation theorem we can find polynomials p_n that converge to f uniformly on $[0,1]$. Uniform convergence on a bounded interval allows the interchange of limit and integral, so we have

$$0 = \lim_{n\to\infty} \int_0^1 p_n(x)f(x)\,dx = \int_0^1 f(x)^2\,dx.$$

But f^2 is continuous, nonnegative, and integrates to 0. So $f \equiv 0$. □

9. Prove that $\int_0^\infty \sin(x^2)\,dx$ converges.

Proof. It suffices to show that the integral on $[1,\infty)$ converges. A substitution of $u = x^2$ gives

$$\int_1^\infty \frac{\sin u}{2\sqrt{u}}\,du.$$

Let $b > 1$. Integrating by parts gives

$$\int_1^b \frac{\sin u}{2\sqrt{u}}\,du = \frac{\cos 1}{2} - \frac{\cos b}{2\sqrt{b}} - \int_1^b \frac{\cos x}{4u^{3/2}}\,dx.$$

This last integral converges absolutely as $b \to \infty$:

$$\int_1^\infty \left|\frac{\cos x}{4u^{3/2}}\right|\,dx \leq \int_1^\infty \frac{1}{4x^{3/2}}\,dx < \infty,$$

and $\cos b/2\sqrt{b} \to 0$ as $b \to \infty$. Therefore taking $b \to \infty$ gives

$$\int_1^\infty \sin(x^2)\,dx = \int_1^\infty \frac{\sin u}{2\sqrt{u}}\,du = \frac{\cos 1}{2} - \int_1^\infty \frac{\cos x}{4u^{3/2}}\,dx,$$

and the integral converges. □

10. Evaluate the limits

(a) $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$

(b) $\lim_{n\to\infty} \left(\frac{1}{n}\sum_{k=1}^{n} (\ln k)^2 - \left(\frac{1}{n}\sum_{k=1}^{n} \ln k \right)^2 \right)$

Proof. (a) Let $L = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$. Then

$$\ln L = \lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} \ln k\right) - \ln n = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \ln(k/n) = \int_0^1 \ln x\, dx = -1,$$

so $L = 1/e$.

(b) Note that

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} (\ln k)^2 &= \frac{1}{n}\sum_{k=1}^{n} \left(\ln^2(k/n) + 2\ln(k/n)\ln n + \ln^2 n\right) \\ &= \frac{1}{n}\sum_{k=1}^{n} \ln^2(k/n) + \frac{2\ln n}{n}\sum_{k=1}^{n} \ln(k/n) + \ln^2 n.\end{aligned}$$

Also notice that

$$\begin{aligned}\ln^2 n - \left(\frac{1}{n}\sum_{k=1}^{n} \ln k\right)^2 &= \left(\ln n + \frac{1}{n}\sum_{k=1}^{n} \ln k\right)\left(\ln n - \frac{1}{n}\sum_{k=1}^{n} \ln k\right) \\ &= \left(2\ln n + \frac{1}{n}\sum_{k=1}^{n} \ln(k/n)\right)\left(-\frac{1}{n}\sum_{k=1}^{n} \ln(k/n)\right) \\ &= -\frac{2\ln n}{n}\sum_{k=1}^{n} \ln(k/n) - \left(\frac{1}{n}\sum_{k=1}^{n} \ln(k/n)\right)^2\end{aligned}$$

Whew. Altogether we have

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} (\ln k)^2 - \left(\frac{1}{n}\sum_{k=1}^{n} \ln k\right)^2\right) &= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} \ln^2(k/n) - \left(\frac{1}{n}\sum_{k=1}^{n} \ln(k/n)\right)^2\right) \\ &= \int_0^1 \ln^2 x\, dx - \left(\int_0^1 \ln x\, dx\right)^2 \\ &= 2 - (-1)^2,\end{aligned}$$

so the limit is 1.

□

11. Define the Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}$$

Prove the following:

(a) The function f is discontinuous at every point.

(b) The function f is not Riemann integrable on any bounded interval.

Proof. (a) Given a rational x find irrationals $x_n \to x$. Then $f(x_n) \to 0 \neq 1 = f(x)$. Similarly, given irrational x find rationals $x_n \to x$. Then $f(x_n) \to 1 \neq 0 = f(x)$. Hence f is continuous nowhere.

(b) Given an interval $[a, b]$ (with $a < b$) and any partition $x_0 < x_1 < \cdots < x_n$ thereof, consider the Riemann sum

$$S = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}),$$

where $c_k \in (x_{k-1}, x_k)$. If we take each c_k rational, then the sum telescopes and $S = b - a$. If we take each c_k irrational then $S = 0$. No matter how small the mesh, there exist two Riemann sums which differ by $b - a > 0$. Thus f is not integrable.

□

12. Define the Riemann ruler function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1/q & \text{if } x = p/q \text{ with } p, q \text{ relatively prime integers and } q > 0 \end{cases}$$

Prove the following:

(a) The function f is continuous at the irrationals and discontinuous at the rationals.

(b) The function f is Riemann integrable on every bounded interval.

(c) For any $a < b$ we have

$$\int_a^b f(x)\,dx = 0.$$

Proof. (a) Let x be rational and find irrationals $x_n \to x$. Then $f(x_n) \to 0 \neq f(x)$. Hence f is not continuous at any rational. Let x be irrational and fix $\epsilon > 0$. Define the set S to be those rationals $y \in (x-1, x+1)$ with denominator less than $1/\epsilon$. Since S is finite, there is $\delta > 0$ so that every point in S is at least δ distance from x. Whenever $y \in \mathbb{R}$ satisfies $|x - y| < \delta$, we have $y \notin S$ and hence $|f(x) - f(y)| < \epsilon$. Therefore f is continuous at x.

(b) Fix a bounded interval I, let $\epsilon > 0$ and consider the set S of rationals in I whose denomiators are less than $|I|/\epsilon$. Note that S is finite and that throughout $I \setminus S$ we have $f \leq \epsilon/|I|$. Given any partition of I, we can refine it to include the set S. Any Riemann sum then has the form

$$\left| \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) \right| \leq \frac{\epsilon}{|I|} \sum_{k=1}^{n} (x_k - x_{k-1}) = \epsilon.$$

This shows that f is Riemann integrable and that the integral is zero. (Remark: the Lebesgue condition for Riemann integrability is that f is integrable if and only if its set of discontinuities has measure zero. The ruler function has a countable number of discontinuities and hence is integrable)

(c) (see part b)

□

13. A continuous function $K : [0,1] \times [0,1] \to \mathbb{R}$ satisfies $|K(x,y)| < 1$ for all $(x,y) \in [0,1] \times [0,1]$. Prove there is a unique continuous $f : \mathbb{R} \to \mathbb{R}$ so that

$$f(x) + \int_0^1 K(x,y) f(y)\,dy = e^x.$$

Proof. Define the linear operator $T : C[0,1] \to C[0,1]$ by

$$(Tf)(x) = e^x - \int_0^1 K(x,y) f(y)\,dy.$$

Note that K is continuous on the compact set $[0,1]^2$; therefore there exists a $k < 1$ so that $|K(x,y)| \leq k$ everywhere. So if we take any two functions $f, g \in C[0,1]$,

$$\|Tf - Tg\|_\infty = \left\| \int_0^1 K(x,y)(f(y) - g(y))\,dy \right\|_\infty \leq k\|f - g\|_\infty.$$

Thus T is a contraction. Since $C[0,1]$ is complete, T has a unique fixed point. □

14. Let (f_n) be a sequence of real-valued uniformly bounded equicontinuous functions on a metric space X. If we define
$$g_n(x) = \max\{f_1(x), f_2(x), \dots, f_n(x)\},$$
prove that the sequence (g_n) converges uniformly.

Proof. The problem is incorrect as written. Here's a counterexample. Let $X = \mathbb{R}$ and define $f_0(x)$ to be a continuous function which is 1 on $(-\infty, 0]$, zero on $[1, \infty)$, and linear on $[1, 2]$. Define $f_n(x) = f_0(x - n)$. Since f_0 is uniformly continuous, the sequence of translates (f_n) is equicontinuous. Further, the sequence is uniformly bounded, with each f_n taking values in $[0, 1]$. Note that $g_n = \max\{f_1 \dots, f_n\} = f_n \to 1$ pointwise, but not uniformly; one can find points $x \in \mathbb{R}$ for which $(g_n(x))$ stays zero for an arbitrarily long time before converging to 1.

Now let's assume that X is compact. We claim that (g_n) is an equicontinuous sequence of functions on X. Let $\epsilon > 0$ and find $\delta > 0$ so that whenever $d(x, y) < \delta$ we have $|f_n(x) - f_n(y)| < \epsilon$ for any n. Suppose $d(x, y) < \delta$ and let $n \in \mathbb{N}$ be arbitrary. Without loss of generality assume $g_n(x) \geq g_n(y)$ and find $k \leq n$ so that $g_n(x) = f_k(x)$. Then $g_n(y) \geq f_k(y)$ and
$$|g_n(x) - g_n(y)| = g_n(x) - g_n(y) = f_k(x) - g_n(y) \leq f_k(x) - f_k(y) < \epsilon.$$
This proves the claim. Next note that (f_n) is uniformly bounded, whence $g(x) = \sup_n f_n(x)$ is defined and bounded on X. Clearly, $g_1 \leq g_2 \leq \cdots \leq g$ and $g_n \to g$ pointwise. This implies that (g_n) is a uniformly bounded equicontinuous family, hence it has a uniformly convergent subsequence by the Arzelá-Ascoli theorem. This subsequence must uniformly converge to the pointwise limit g. We can show that the entire sequence converges uniformly to g. Let $\epsilon > 0$ and find $k > 0$ so that $\|g_{n_k} - g\|_\infty < \epsilon$. For all $n \geq n_k$ we have $g_{n_k} \leq g_n \leq g$, so
$$\|g_n - g\|_\infty \leq \|g_{n_k} - g\|_\infty < \epsilon,$$
proving uniform convergence, as desired. □

15. Suppose that K is a nonempty compact subset of a metric space X. Given $x \in X$ prove there exists a point $z \in K$ so that
$$d(x, z) = \operatorname{dist}(x, K).$$

Proof. Fix $x \in X$ and find a sequence (z_n) in K so that $d(x, z_n) \to \operatorname{dist}(x, K)$. Since K is compact there is a convergent subsequence $z_{n_k} \to z \in K$. Then we have
$$\operatorname{dist}(x, K) = \lim_{k\to\infty} d(x, z_{n_k}) = d(x, z).$$
□

16. Prove every compact metric space has a countable dense subset.

Proof. We only need total boundedness of the space. For each $n \in \mathbb{N}$ find a finite $(1/n)$-net B_n. The set $\cup_n B_n$ is countable (since each B_n is finite). Furthermore, it is dense; given a point x in the space and $\epsilon > 0$, we can find $n > 1/\epsilon$ and a point in B_n which is at most ϵ away from x. □

17. Evaluate $\displaystyle\lim_{n\to\infty} \int_0^1 \left(1 + \frac{x}{n}\right)^n dx$ with justification.

Proof. The functions $f_n(x) = (1 + x/n)^n$ converge pointwise to the continuous function $f(x) = e^x$ on $[0, 1]$. Furthermore, it can be shown that the sequence (f_n) is monotone. Dini's theorem implies that the convergence is uniform, so we can take the limit inside of the integral:
$$\lim_{n\to\infty} \int_0^1 \left(1 + \frac{x}{n}\right)^n dx = \int_0^1 e^x\, dx = e - 1.$$
□

18. Show that

$$\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \cdots$$

converges uniformly on $\mathbb{R}$ but never absolutely.

Proof. For each fixed x the series is an alternating harmonic series, which converges, but not absolutely. Since the terms are monotonically decreasing to zero in absolute value, the alternating series test tells us that

$$\left|\sum_{k=n}^{\infty} \frac{(-1)^{k-1}}{k+x^2}\right| \leq \frac{1}{n+x^2} \leq \frac{1}{n}.$$

Since the tail of the series converges to zero at a rate independent of x, the convergence is uniform. □

19. (a) Prove the polynomials of even degree are dense in the space of continuous functions $C[0,1]$.
    (b) Is this still true on $C[-1,1]$?

Proof. (a) Fix $f \in C[0,1]$. On $[0,1]$ the function $\sqrt{x}$ is continuous, so for any $\epsilon > 0$ we can find a polynomial p with $|p(x) - f(\sqrt{x})| < \epsilon$ throughout $[0,1]$. That is, $|p(x^2) - f(x)| < \epsilon$ throughout $[0,1]$. The polynomial $p(x^2)$ is even, so we are done.

(b) No, it is not true. Suppose there is an even polynomial p with $|p(x) - x| < 1$ throughout $[-1,1]$. Then $0 < p(1) < 2$, yet $-2 < p(-1) < 0$. An even polynomial must have $p(-1) = p(1)$, so such a p cannot exist.

□

20. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is integrable on every interval of the form $[0, A]$ for $A > 0$ and that $f \to 1$ as $x \to \infty$. Prove that

$$\lim_{s \to 0^+} s \int_0^\infty e^{-st} f(t)\, dt = 1.$$

Proof. Note that

$$\lim_{s \to 0^+} s \int_0^\infty e^{-st} f(t)\, dt = \lim_{s \to 0^+} s \int_0^\infty e^{-st} (f(t) - 1)\, dt + 1,$$

so if we instead assume that $f \to 0$ as $x \to \infty$, then we wish to show the limit is zero.

Let $\epsilon > 0$. Find $A > 0$ so that for all $x \geq A$ we have $|f(x)| \leq \epsilon$. On one hand,

$$\left| s \int_A^\infty e^{-st} f(t)\, dt \right| \leq \epsilon \int_A^\infty s e^{-st}\, dt = \epsilon e^{-sA} < \epsilon.$$

On the other hand,

$$\left| s \int_0^A e^{-st} f(t)\, dt \right| \leq s \int_0^A f(t)\, dt = C_\epsilon s$$

for some constant C_ϵ depending only on ϵ. Therefore

$$\limsup_{s \to 0^+} \left| s \int_0^\infty e^{-st} f(t)\, dt \right| \leq \limsup_{s \to 0^+} (C_\epsilon s + \epsilon) = \epsilon.$$

Since ϵ was arbitrary, the limit is zero, as desired. □

21. Define, for $x, y > 1$

$$f(x,y) = \frac{x-y}{1-xy}.$$

For each fixed y, note that $f(x,y) \to 1$ as $x \to 1$. Is the convergence uniform in y?

Proof. Suppose the convergence were uniform in y. Then we can find $\delta > 0$ so that for any $y > 1$ and $1 < x < 1 + \delta$ we have

$$\left| \frac{x-y}{1-xy} - 1 \right| < 1.$$

Since y is arbitrary, we can take $y \to 1$ to find

$$\left| \frac{x-1}{1-x} - 1 \right| \leq 1,$$

or $2 \leq 1$. This contradiction implies the convergence is not uniform. □

22. (a) Suppose that (a_{nk}) is a doubly-indexed series of positive terms. Prove that

$$\sum_k \sum_n a_{nk} = \sum_n \sum_k a_{nk},$$

where ∞ is allowed.

(b) Give an example of a sequence for which the above equation fails.

Proof. (a) Denote $S_1 = \sum_k \sum_n a_{nk}$ and $S_2 = \sum_n \sum_k a_{nk}$. A sum $\sum_n a_{nk}$ at constant k is called a row sum, while a sum $\sum_k a_{nk}$ at constant n is a column sum. With this terminology we can proceed.

To begin, assume that a single row or column sum is divergent. Without loss of generality, say $\sum_k a_{Nk} = \infty$ for some N. Then certainly $S_2 = \infty$ as well. Given $M > 0$ truncate the series so that

$$\sum_{k=1}^{K} a_{Nk} \geq M.$$

Then we have that

$$S_1 = \sum_k \sum_n a_{nk} \geq \sum_{k=1}^{K} \sum_{n=1}^{N} a_{nk} \geq \sum_{k=1}^{K} a_{Nk} \geq M.$$

Since M is arbitrary, $S_1 = \infty$ as well.

Next, we assume that each row and column is a convergent series, but either S_1 or S_2 is infinite anyway. Without loss of generality, say $S_1 = \infty$. Given $M > 0$ find K so that

$$\sum_{k=1}^{K} \sum_n a_{nk} \geq M.$$

Since each row sum converges, we can truncate each to within $1/K$ of its full sum; that is, find N so that

$$\sum_n a_{nk} - \sum_{n=1}^{N} a_{nk} \leq \frac{1}{K}.$$

for each $k = 1, 2, \ldots, K$. It follows that

$$S_2 = \sum_n \sum_k a_{nk} \geq \sum_{n=1}^{N} \sum_k a_{nk} \geq \sum_{n=1}^{N} \sum_{k=1}^{K} a_{nk} = \sum_{k=1}^{K} \sum_{n=1}^{N} a_{nk} \geq M - 1.$$

Since M is arbitrary, $S_2 = \infty$ as well.

Finally, we assume that S_1 and S_2 are both finite. Let $\epsilon > 0$ and find K_1 so that

$$S_1 - \sum_{k=1}^{K_1} \sum_n a_{nk} < \epsilon.$$

Since each row sum converges, we can truncate them to within ϵ/K_1; that is, find N_1 so that

$$\sum_{n=1}^{N_1} a_{nk} < \frac{\epsilon}{K_1}$$

for each $k = 1, 2 \dots, K_1$. Then

$$S_1 - \sum_{k=1}^{K_1} \sum_{n=1}^{N_1} a_{nk} < 2\epsilon.$$

Notice that this relation is preserved if we increase either K_1 or N_1 since all terms are positive. Similarly we can find K_2, N_2 so that

$$S_2 - \sum_{n=1}^{N_2} \sum_{k=1}^{K_2} a_{nk} < 2\epsilon.$$

Now let $K' = \max\{K_1, K_2\}$ and $N' = \max\{N_1, N_2\}$ to find that

$$S_j - \sum_{n=1}^{N'} \sum_{k=1}^{K'} a_{nk} < 2\epsilon$$

for both $j = 1, 2$. Hence $|S_1 - S_2| < 4\epsilon$. As ϵ was arbitrary, we must have $S_1 = S_2$, as desired.

(b) For $n \geq 0$ let $a_{nn} = 1$, $a_{n(n+1)} = -1$, and all other $a_{nk} = 0$. Then

$$\sum_n a_{nk} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{else} \end{cases}$$

and $\sum_k a_{nk} = 0$ for all n. Therefore

$$\sum_k \sum_n a_{nk} = 1 \neq 0 = \sum_n \sum_k a_{nk}.$$

□

23. Let $K : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous. Define $T : C[0, 1] \to C[0, 1]$ to be the linear operator

$$Tf(x) = \int_0^1 K(x, y) f(y)\, dy.$$

Prove that T maps bounded subsets of $C[0, 1]$ into precompact ones.

Proof. By Arzelá-Ascoli it suffices to show that T maps uniformly bounded sets to uniformly bounded and equicontinuous ones. Assume that $S \subset C[0, 1]$ is uniformly bounded and $M > 0$ is chosen so that $\|f\|_\infty \leq M$ for all $f \in S$. Since K is continuous on the compact set $[0, 1]^2$ there is a constant A so that $|K| \leq A$ everywhere. Thus

$$\|Tf(x)\|_\infty = \left\| \int_0^1 K(x, y) f(y)\, dy \right\|_\infty \leq AM,$$

so that $T(S)$ is uniformly bounded. Now let $\epsilon > 0$. The continuous function K has a compact domain, hence is uniformly continuous. Find $\delta > 0$ so that for any $|x_1 - x_2| < \delta$ and any y we have

$$|K(x_1, y) - K(x_2, y)| < \epsilon / M.$$

Given any $f \in S$ and $|x_1 - x_2| < \delta$ we have

$$|Tf(x_1) - Tf(x_2)| = \left| \int_0^1 (K(x_1, y) - K(x_2, y)) f(y)\, dy \right| \leq \frac{\epsilon}{M} \left| \int_0^1 f(y)\, dy \right| \leq \epsilon.$$

So the family $T(S)$ is equicontinuous, as desired. □

24. (a) Let f be a continuous periodic function with some period t. Show that its set of translates

$$\mathcal{F} = \{f(x - t) : t \in \mathbb{R}\}$$

is compact in $C(\mathbb{R})$.

(b) A function is called almost periodic if its set of translates is precompact. Prove the set of almost periodic functions is a closed subalgebra of $C(\mathbb{R})$.

Proof. (a) Given a sequence $(f(x-a_n))$ of translates we wish to find a convergent subsequence. Note that we can reduce each a_n modulo t and assume each $a_n \in [0,t]$. Since $[0,t]$ is compact, there is a convergent subsequence (a_{n_k}) converging to some $a \in [0,t]$.

Fix $\epsilon > 0$. Note that f is uniformly continuous on $[0,t]$ and hence on $\mathbb{R}$; thus we can find $\delta > 0$ so that whenever $|x-y| < \delta$ we have $|f(x)-f(y)| < \epsilon$. Find N so that for any $n \geq N$ it follows that $|a_{n_k} - a| < \delta$. For all such n and arbitrary $x \in \mathbb{R}$,

$$|f(x-a_{n_k}) - f(x-a)| < \epsilon.$$

This proves that $f(x-a_{n_k}) \to f(x-a)$ uniformly as $k \to \infty$.

(b) First, a word of warning/clarification: the topology of $C(\mathbb{R})$ is more complicated than that of $C[0,1]$ since continuous functions on $\mathbb{R}$ need not have finite supremum norms. We call $C(\mathbb{R})$ a Fréchet space—its topology is generated by a family of "semi-norms." Here's all we need: define $f_n \to f$ in $C(\mathbb{R})$ to mean $\sup_K |f_n(x) - f(x)| \to 0$ for every compact set $K \subset \mathbb{R}$.

We need to prove that the sum and product of almost periodic functions are almost periodic and that almost periodicity is preserved by uniform convergence on compact sets. Given almost periodic functions $f, g \in C(\mathbb{R})$ define

$$\mathcal{F}(f) = \{f(x-t) : t \in \mathbb{R}\}$$

and similarly for $\mathcal{F}(g)$. We need to show that a sequence in $\mathcal{F}(f+g)$ has a Cauchy subsequence—that is, a subsequence which is uniformly Cauchy in $C(K)$ for any compact set $K \subset \mathbb{R}$. Given a sequence $(f_n + g_n)$ in $\mathcal{F}(f+g)$, pass to a subsequence $(f_{n_k} + g_{n_k})$ so that (f_{n_k}) is Cauchy. Pass to a smaller subsequence so that (g_{n_k}) is Cauchy as well. On any compact set,

$$\|f_{n_k} + g_{n_k} - f_{n_j} - g_{n_j}\|_\infty \leq \|f_{n_k} - f_{n_j}\|_\infty + \|g_{n_k} - g_{n_j}\|_\infty \to 0$$

as $k, j \to \infty$. Thus $(f_{n_k} + g_{n_k})$ is Cauchy.

Now assume that $(f_n g_n)$ is a sequence in $\mathcal{F}(fg)$; we wish to find a Cauchy subsequence. As before, pass to a subsequence so that each of (f_{n_k}) and (g_{n_k}) is Cauchy. Then on any compact set

$$\begin{aligned}\|f_{n_k} g_{n_k} - f_{n_j} g_{n_j}\|_\infty &\leq \|f_{n_k} g_{n_k} - f_{n_k} g_{n_j}\|_\infty + \|f_{n_k} g_{n_j} - f_{n_j} g_{n_j}\|_\infty \\ &\leq \|g_{n_k} - g_{n_j}\|_\infty \|f_{n_k}\|_\infty + \|f_{n_k} - f_{n_j}\|_\infty \|g_{n_j}\|_\infty.\end{aligned}$$

Recall that Cauchy sequences are bounded; thus we can find $M > 0$ so that $\|f_{n_k}\|_\infty, \|g_{n_j}\|_\infty \leq M$ for all k, j. Therefore

$$\|f_{n_k} g_{n_k} - f_{n_j} g_{n_j}\|_\infty \leq M\|g_{n_k} - g_{n_j}\|_\infty + M\|f_{n_k} - f_{n_j}\|_\infty \to 0$$

as $k, j \to \infty$. We conclude that $(f_{n_k} g_{n_k})$ is a Cauchy subsequence.

Finally, we need to show that almost periodicity is preserved under uniform convergence on compact sets. We need to be careful with notation; let $f_n \to f$ in the aforementioned sense, with each f_n almost periodic. Consider a sequence $(f(x-t_n))$ of translates of f. We wish to find a Cauchy subsequence. Consider instead the sequence $(f_1(x-t_n))$; we can pass to a Cauchy subsequence. Take the first term of the subsequence of (t_n) and denote it s_1. Passing to a further subsequence (still with s_1 at the front) we can assume that $(f_2(x-t_n))$ is Cauchy as well. Denote the second term of this subsequence by s_2. Proceed inductively this way, building a subsequence $s_1, s_2, s_3, \ldots$ of (t_n). For each k, we see that $(f_k(x-s_n))$ is a Cauchy sequence; we'll show that $(f(x-s_n))$ is Cauchy as well.

Let $\epsilon > 0$ and fix an underlying compact set. Find n so that $\|f_n - f\|_\infty < \epsilon/3$. Find $M > 0$ so that

$$\|f_n(x-s_k) - f_n(x-s_j)\|_\infty < \epsilon/3$$

for all $k, j \geq M$. For all such k, j we have

$$\begin{aligned}&\|f(x-s_k) - f(x-s_j)\|_\infty \\ &\leq \|f(x-s_k) - f_n(x-s_k)\|_\infty + \|f_n(x-s_k) - f_n(x-s_j)\|_\infty + \|f_n(x-s_j) - f(x-s_j)\|_\infty \\ &< \epsilon.\end{aligned}$$

Thus we've found a Cauchy subsequence and are done.

□